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On some Applications of the Units of an n-fold Space.

BY C. H. CHAPMAN.

The following article originated in an attempt to obtain a proof of the rule for multiplying two determinants of the n^{th} order by the principles of quaternions. There is no difficulty, of course, in the case of a determinant of the third order, but beyond that it seemed necessary to use a more general system of vector units than Hamilton's i, j, k. The symmetry of an n-dimensional space, where n is odd, leads at once to the proper assumptions, as has been often shown. In working with n rectangular unit vectors the symbols S and V as defined by Hamilton were found not sufficiently exclusive, and, as a makeshift, a new symbol ω_κ was defined and found extremely useful. When applied to the product of two vectors ω_κ has the same selective power as V, but applied to the product of three vectors, it selects the same terms as S. Applied to a product of more than three vectors ω_κ is equivalent neither to S nor V.

Aided by this symbol, the rules for determinants of the n^{th} order were found with ease and a method of inverting the linear and vector function ϕ in space of n dimensions was generalized from that given by Hamilton for ordinary space. The fact that $\phi\rho$ effects a linear substitution upon the coefficients of ρ leads on to the interesting connection between the operator ϕ and the theory of linear differential equations, and by means of its properties a number of theorems may be demonstrated with facility.

I begin by showing the connection between ordinary quaternions and determinants of the third order.

If α, β, γ and α', β', γ' be two systems of non-coplanar vectors in space of three dimensions, it is easy to obtain the following equation, viz:

$$S\alpha\beta\gamma \,.\, S^3\alpha'\beta'\gamma' = S\,.\,V\alpha'\beta'\,V\beta'\gamma'\,V\gamma'\alpha' \begin{vmatrix} S\alpha'\alpha\,, & S\beta'\alpha\,, & S\gamma'\alpha \\ S\alpha'\beta\,, & S\beta'\beta\,, & S\gamma'\beta \\ S\alpha'\gamma\,, & S\beta'\gamma\,, & S\gamma'\gamma \end{vmatrix}$$

But plainly,

$$\left.\begin{aligned} S.V\alpha'\beta'V\beta'\gamma'V\gamma'\alpha' &= S.V\alpha'\beta'V.V\beta'\gamma'V\gamma'\alpha' \\ &= -S.V\alpha'\beta'.\gamma'S\alpha'\beta'\gamma' = -S^2\alpha'\beta'\gamma'. \end{aligned}\right\} \qquad (1)$$

Hence

$$S\alpha\beta\gamma.S\alpha'\beta'\gamma' = -\begin{vmatrix} S\alpha'\alpha, & S\beta'\alpha, & S\gamma'\alpha \\ S\alpha'\beta, & S\beta'\beta, & S\gamma'\beta \\ S\alpha'\gamma, & S\beta'\gamma, & S\gamma'\gamma \end{vmatrix} \qquad (2)$$

If now

$$\alpha = x_1 i + y_1 j + z_1 k,\ \beta = x_2 i + y_2 j + z_2 k,\ \gamma = x_3 i + y_3 j + z_3 k,$$
$$\alpha' = x_1' i + y_1' j + z_1' k,\ \beta' = x_2' i + y_2' j + z_2' k,\ \gamma' = x_3' i + y_3' j + z_3' k,$$

then

$$S\alpha\beta\gamma = -\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix},\ -S\alpha'\beta'\gamma' = \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_3', & y_3', & z_3' \end{vmatrix}$$

and

$$\begin{aligned} S\alpha'\alpha &= -(x_1x_1' + y_1y_1' + z_1z_1'), \\ S\beta'\alpha &= -(x_1x_2' + y_1y_2' + z_1z_2'), \\ S\gamma'\alpha &= -(x_1x_3' + y_1y_3' + z_1z_3'), \end{aligned}$$

and so on for the other scalars. Hence

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \times \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_3', & y_3', & z_3' \end{vmatrix} = \begin{vmatrix} x_1x_1'+y_1y_1'+z_1z_1', & x_1x_2'+y_1y_2'+z_1z_2', & x_1x_3'+y_1y_3'+z_1z_3' \\ x_2x_1'+y_2y_1'+z_2z_1', & x_2x_2'+y_2y_2'+z_2z_2', & x_2x_3'+y_2y_3'+z_2z_3' \\ x_3x_1'+y_3y_1'+z_3z_1', & x_3x_2'+y_3y_2'+z_3z_2', & x_3x_3'+y_3y_3'+z_3z_3' \end{vmatrix}$$

by eq. (2). Giving thus the rule for multiplying determinants of the third order. Noting that $V\alpha\beta = (x_1y_2 - x_2y_1)\,ij + \ldots\ldots = (x_1y_2 - x_2y_1)\,k + \ldots\ldots$, eq. (1) gives

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}^2 = \begin{vmatrix} x_1y_2 - y_1x_2, & z_1x_2 - x_1z_2, & y_1z_2 - z_1y_2 \\ x_2y_3 - y_2x_3, & z_2x_3 - x_2z_3, & y_2z_3 - z_2y_3 \\ x_3y_1 - y_3x_1, & z_3x_1 - x_3z_1, & y_3z_1 - z_3y_1 \end{vmatrix}$$

or the square of a determinant of the third order is equal to the determinant of its minors.

To generalize these results I consider a system of n units like i, j, k in a space of n dimensions, where n is an odd integer. Let these units be $l_1, l_2, l_3, \ldots\ldots l_n$, and let $Sl_\kappa l_\tau = 0$; also $l_\kappa l_\tau = -l_\tau l_\kappa$. Also the complete cycle when multiplied together produces a scalar. Assume then

$$l_1 l_2 \ldots\ldots l_{n-1} l_n = A \qquad (3)$$

where $A = \pm 1$, as will be shown.

Again, since $l^2 = ll = -1$, it follows that

$$l_\kappa = -\frac{1}{l_\kappa} = -l_\kappa^{-1}.$$

To determine the rule for the essential sign of A, note that, by eqs. (3) and (4),
$$l_1 l_2 \dots\dots l_{n-1} = -Al_n.$$
Also by successive multiplications
$$l_n = Al_{n-1}^{-1} l_{n-2}^{-1} \dots\dots l_1^{-1} = (-1)^{n-1} A l_{n-1} l_{n-2} \dots\dots l_1 = A l_{n-1} l_{n-2} \dots\dots l_1,$$
since n is odd. By the continued use of eq. (4) this becomes
$$l_n = (-1)^{\frac{(n-1)(n-2)}{2}} A . l_1 l_2 l_3 \dots\dots l_{n-1},$$
or
$$l_1 l_2 \dots\dots l_{n-1} = (-1)^{\frac{(n-1)(n-2)}{2}} \frac{1}{A};$$
hence
$$-A = (-1)^{\frac{(n-1)(n-2)}{2}} \frac{1}{A},$$
or
$$A = (-1)^{\frac{n(n-1)}{2}} \frac{1}{A}. \tag{5}$$
This is satisfied only if $A = (-1)^{\frac{n(n-1)}{2}}$. From the symmetry of the space it is safe to conclude that in general
$$l_\kappa l_{\kappa+1} \dots\dots l_n l_1 l_2 \dots\dots l_{\kappa-1} = (-1)^{\frac{n(n-1)}{2}} = A,$$
or if proof were needed, multiplying l_1 by the first member of eq. (3), then multiplying the result by l_1, we have, if the equation be preserved,
$$l_1 l_1 l_2 l_3 \dots\dots l_n l_1 = A l_1 l_1,$$
or
$$-l_2 l_3 \dots\dots l_n l_1 = -A,$$
or
$$l_2 l_3 \dots\dots l_n l_1 = A,$$
a process which may be carried on indefinitely. Thus both the sign and numerical value of A are consistently determined by eq. (5) for the whole series of cyclic products formed from the n units.

To determine what the value of $l_1 l_2$ is, since $Sl_1 l_2 = 0$, it may be assumed at once that
$$l_1 l_2 = x_1 l_1 + x_2 l_2 + x_3 l_3 + \dots\dots + x_n l_n.$$
Then
$$Sl_1 l_1 l_2 = 0 = -x_1 + x_2 S l_1 l_2 + \dots\dots = -x_1.$$
Likewise
$$x_2 = 0 \text{ and } x_3 = -Sl_1 l_2 l_3,\ x_4 = -Sl_1 l_2 l_4, \dots\dots$$
Hence the product of two vectors is a linear function of the remaining $n-2$ of the system.

Let it be required to express a vector ρ linearly in terms of the fundamental units $l_1 l_2 \dots\dots l_n$.

Assume the equation

$$\rho = x_1 l_1 + x_2 l_2 + \ldots\ldots + x_n l_n .$$

Then at once, multiplying by l_μ and taking scalars,

$$x_\mu = - S l_\mu \rho , \tag{6}$$

which determines the coefficients. If

$$\rho_1 = y_1 l_1 + y_2 l_2 + \ldots\ldots + y_n l_n ,$$

since $l_\kappa l_\tau = - l_\tau l_\kappa$,

$$V\rho\rho_1 = - V\rho_1\rho , \tag{7}$$

and plainly also, $S\rho\rho_1 = S\rho_1\rho$.

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ be a system of n vectors such that

$$\left.\begin{array}{l} \alpha_1 = x_{11} l_1 + x_{12} l_2 + x_{13} l_3 + \ldots\ldots + x_{1n} l_n , \\ \alpha_2 = x_{21} l_1 + x_{22} l_2 + x_{23} l_3 + \ldots\ldots + x_{2n} l_n , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n = x_{n1} l_1 + x_{n2} l_2 + x_{n3} l_3 + \ldots\ldots + x_{nn} l_n . \end{array}\right\} \tag{A}$$

Let ω_κ be a symbol such that when applied to the product of any κ vectors it shall select all those terms containing the product of any κ distinct units and shall exclude all other terms, the selected terms being united into an aggregate by addition.

Let $\omega_\kappa \alpha_1 \alpha_2 \ldots\ldots \alpha_\kappa$, or briefly $\omega_\kappa(\alpha_1)$, when the cyclic order is perfect, be such an aggregate. Considering any two vectors $\alpha_1\alpha_2$, it is clear that only those terms of their product included in $\omega_2\alpha_1\alpha_2$ enter into ω_κ. Hence the symbol ω_2 may be arbitrarily inserted in a product under ω_κ. And in general, if $\lambda < \kappa$, to insert ω_λ under ω_κ before any group of λ vectors is merely to bring into evidence an operation implied in ω_κ itself. Therefore

$$\omega_\kappa \alpha_1 \ldots\ldots \alpha_\kappa = \omega_\kappa \alpha_1 \alpha_2 \ldots\ldots \alpha_\mu \left[\omega_\lambda \alpha_{\mu+1} \ldots\ldots\right] \alpha_{\mu+\lambda+1} \ldots\ldots \alpha_\kappa .$$

ω_κ being merely selective, is plainly commutative with all scalars and distributive over a sum.

$\omega_\kappa(\alpha_1)$ is the sum of all the determinants in the matrix

$$\left|\begin{array}{l} x_{11},\ x_{12},\ x_{13},\ \ldots\ldots\ x_{1n} \\ x_{21},\ x_{22},\ x_{23},\ \ldots\ldots\ x_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{\kappa 1},\ x_{\kappa 2},\ x_{\kappa 3},\ \ldots\ldots\ x_{\kappa n} \end{array}\right| ,$$

each multiplied by the group of units whose suffixes are determined by the second suffixes of the principal diagonal. So far as the letters in each product are concerned, this is only another statement of the definition of ω_κ; and as to the signs, it need only be noticed that given any group of $\varkappa$ units,

$$(M),\ l_1 l_3 l_4 l_5 \ldots\ldots l_\tau l_\mu$$

where no two are identical, then any arrangement produced from (M) by λ interchanges of consecutive vectors equals $(-1)^\lambda (M)$, so that the signs of the various terms are precisely those required. Hence, taking $\varkappa = n$, $\omega_n(\alpha_1 \ldots\ldots \alpha_n)$

$$= \begin{vmatrix} x_{11}, & x_{12}, & \ldots\ldots & x_{1n} \\ x_{21}, & x_{22}, & \ldots\ldots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}, & x_{n2}, & \ldots\ldots & x_{nn} \end{vmatrix} A,$$

where $A = l_1 \ldots\ldots l_n$.

When dealing with only two vectors, ω_2 is the same as V, as a glance at the system (A) will show; and $\omega_{n-1}\alpha_1 \ldots\ldots \alpha_{n-1}$ is a vector, since the product of any $n-1$ units produces the remaining one multiplied by $\pm A$; but

$$\omega_{n-1}(\alpha_1 \ldots\ldots \alpha_{n-1}) \neq V.\alpha_1 \ldots\ldots \alpha_{n-1}.$$

Also, while $\omega_n \alpha_1 \ldots\ldots \alpha_n$ is a scalar, it is not the same as $S.\alpha_1 \ldots\ldots \alpha_n$.

From eq. (7), then, $\omega_2 \rho_1 \rho = -\omega_2 \rho \rho_1$. Hence

$$\begin{aligned} \omega_n \alpha_1 \ldots\ldots \alpha_{\kappa-1} \alpha_\kappa \ldots\ldots \alpha_n &= \omega_n \alpha_1 \ldots\ldots (\omega_2 \alpha_{\kappa-1} \alpha_\kappa) \ldots\ldots \alpha_n \\ &= -\omega_n \alpha_1 \ldots\ldots (\omega_2 \alpha_\kappa \alpha_{\kappa-1}) \ldots\ldots \alpha_n \\ &= -\omega_n \alpha_1 \ldots\ldots \alpha_\kappa \alpha_{\kappa-1} \ldots\ldots \alpha_n . \end{aligned}$$

Therefore, to interchange two rows (or columns) changes the sign of the determinant.

Again, $\omega_n \alpha_1 \ldots\ldots \alpha_\kappa \alpha_\kappa \ldots\ldots \alpha_{n-1} = \omega_n \alpha_1 \ldots\ldots (\omega_2 \alpha_\kappa \alpha_\kappa) \ldots\ldots \alpha_{n-1}$. But $\omega_2 \alpha_\kappa \alpha_\kappa = 0$ evidently. Hence $\omega_n(\alpha) = 0$ if it contains the same vector twice, and a determinant vanishes if two of its rows are identical.

If one of the n vectors $\alpha_1 \ldots\ldots \alpha_\mu \ldots\ldots \alpha_n$ is a linear function of the others, we have

$$\omega_n \alpha_1 \ldots\ldots \alpha_\mu \ldots\ldots \alpha_n = \omega_n \alpha_1 \alpha_2 \ldots\ldots (c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n) \ldots\ldots \alpha_n = 0,$$

since every product will contain some vector twice. Also

$$\omega_n \alpha_1 \alpha_2 \ldots\ldots (\alpha_\mu + c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots) \ldots\ldots \alpha_n = \omega_n \alpha_1 \ldots\ldots \alpha_n,$$

since the products formed with the added vectors all vanish. Hence a determinant is not changed by adding to the terms of any row the corresponding terms of any other row multiplied by a common multiplier.

Consider the system

$$\beta_1 = y_{11}\alpha_1 + y_{12}\alpha_2 + \dots\dots + y_{1n}\alpha_n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\beta_n = y_{n1}\alpha_1 = y_{n2}\alpha_2 + \dots\dots + y_{nn}\alpha_n.$$

From what has already been shown it is clear that

$$\omega_n\beta_1\beta_2\dots\dots\beta_n = \begin{vmatrix} y_{11}, & y_{12}, & y_{13} & \dots\dots & y_{1n} \\ y_{21}, & y_{22}, & y_{23} & \dots\dots & y_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ y_{n1}, & y_{n2}, & y_{n3} & \dots\dots & y_{nn} \end{vmatrix} \omega_n\alpha_1\dots\dots\alpha_n,$$

which again

$$= \begin{vmatrix} y_{11}, & y_{12} & \dots\dots & y_{1n} \\ \dots & \dots & \dots & \dots \\ y_{n1}, & y_{n2} & \dots\dots & y_{nn} \end{vmatrix} \times \begin{vmatrix} x_{11}, & x_{12} & \dots\dots & x_{1n} \\ \dots & \dots & \dots & \dots \\ x_{n1}, & x_{n2} & \dots\dots & x_{nn} \end{vmatrix} A.$$

But

$$\beta_1 = (y_{11}x_{11} + y_{12}x_{21} + y_{13}x_{31} + \dots\dots + y_{1n}x_{n1})\,l_1$$
$$+ (y_{11}x_{12} + y_{12}x_{22} + y_{13}x_{32} + \dots\dots + y_{1n}x_{n2})\,l_2 + \dots\dots,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\beta_\mu = (y_{\mu1}x_{11} + y_{\mu2}x_{21} + \dots\dots)\,l_1 + (y_{\mu1}x_{12} + y_{\mu2}x_{22} + \dots\dots)\,l_2 + \dots\dots$$

Hence $\omega_n\beta_1\beta_2\dots\dots\beta_n$

$$= \begin{vmatrix} y_{11}x_{11} + y_{12}x_{21} + \dots\dots + y_{1n}x_{n1}, & y_{11}x_{12} + y_{12}x_{22} + \dots\dots + y_{1n}x_{n2}, & \dots\dots, & \dots\dots \\ \dots & \dots & \dots & \dots \\ y_{n1}x_{11} + y_{n2}x_{21} + \dots\dots + y_{nn}x_{n1}, & y_{n1}x_{12} + y_{n2}x_{22} + \dots\dots + y_{nn}x_{n2}, & \dots\dots, & \dots\dots \end{vmatrix} A.$$

These results must be equal since ω_n selects the same terms from a product whatever be the intermediate steps of the work, provided no reductions are made among the units. Thus we have the rule for multiplying determinants. I shall obtain it again in a moment.

Assume a system of functions as follows:

$$h_1 = z_{11}\omega_{n-1}(\alpha_1) + z_{12}\omega_{n-1}(\alpha_2) + \dots\dots + z_{1n}\omega_{n-1}(\alpha_n),$$
$$h_2 = z_{21}\omega_{n-1}(\alpha_1) + z_{22}\omega_{n-2}(\alpha_2) + \dots\dots + z_{2n}\omega_{n-1}(\alpha_n),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$h_n = z_{n1}\omega_{n-1}(\alpha_1) + z_{n2}\omega_{n-2}(\alpha_2) + \dots\dots + z_{nn}\omega_{n-1}(\alpha_n),$$

where $\omega_{n-1}(\alpha_\mu) = \omega_{n-1}\alpha_\mu\alpha_{\mu+1}\dots\dots\alpha_n\alpha_1\alpha_2\dots\dots\alpha_{\mu-2}$.

Let $\omega_{n-1}^{\lambda}h_1h_2\dots\dots h_\lambda$ designate the result obtained by taking out of the product $h_1h_2\dots\dots h_\lambda$ all those terms containing the product of λ distinct cyclic groups of units such as $l_\mu l_{\mu+1}\dots\dots l_n l_1 l_2\dots\dots l_{\mu-1}$. ω_{n-1}^{λ} is distributive over a sum, being merely selective; hence the products obtained may be separately considered. The same reasoning which showed that

$$\omega_\kappa \alpha_1 \alpha_2 (\omega_\mu \alpha_3 \alpha_4 \ldots\ldots) \ldots\ldots \alpha_\kappa = \omega_\kappa \alpha_1 \alpha_2 \ldots\ldots \alpha_\kappa$$

shows that

$$\omega_{n-1}^\lambda h_1 h_2 \ldots\ldots (\omega_{n-1}^\delta h_\kappa h_{\kappa+1} \ldots\ldots) \ldots\ldots h_\lambda ,$$

where ω_{n-1}^δ operates on any δ functions, $\delta < \lambda$.

In the first place,

$$\omega_{n-1}^2 [\omega_{n-1}(\alpha_1)\,\omega_{(n-1)}(\alpha_1)] = 0 .$$

For

$$\begin{aligned} \omega_{n-1}(\alpha_1) &= C_n l_1 l_2 \ldots\ldots l_{n-1} + C_1 l_2 l_3 \ldots\ldots l_n + \ldots\ldots \\ &= -A\,[C_n l_n + C_1 l_1 + C_2 l_2 + C_3 l_3 + \ldots\ldots] \end{aligned}$$

Hence

$$\omega_{n-1}^2 [\omega_{n-1}(\alpha_1)\,\omega_{n-1}(\alpha_1)] = A^2 [(C_n C_1 - C_1 C_n)\, l_n l_1 + \ldots\ldots] = 0 .$$

Also

$$\omega_{n-1}^2 [\omega_{n-1}(\alpha_\lambda)\,\omega_{n-1}(\alpha_\mu)] = -\,\omega_{n-1}^2 [\omega_{n-1}(\alpha_\mu)\,\omega_{n-1}(\alpha_\lambda)] ;$$

for both are linear in the n units. This gives the rule for the signs and shows that no function $\omega_{n-1}(\alpha_\mu)$ can come in twice under ω_{n-1}^λ in the same product. It follows just as in previous cases that

$$\omega_{n-1}^n h_1 h_2 \ldots\ldots h_n = \begin{vmatrix} z_{11}, & z_{12}, & z_{13}, & \ldots\ldots & z_{1n} \\ z_{21}, & z_{22}, & z_{23}, & \ldots\ldots & z_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_{n1}, & z_{n2}, & z_{n3}, & \ldots\ldots & z_{nn} \end{vmatrix} \omega_{n-1}^n [\omega_{n-1}(\alpha_1)\,\omega_{n-1}(\alpha_2) \ldots\ldots \omega_{n-1}(\alpha_n)] . \tag{8}$$

As before shown,

$$\left.\begin{aligned} \omega_{n-1}(\alpha_1) &= -A\,[C_{1n} l_n + C_{11} l_1 + C_{12} l_2 + \ldots\ldots + C_{1,\,n-1} l_{n-1}] , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_{n-1}(\alpha_n) &= -A\,[C_{nn} l_n + C_{n1} l_1 + C_{n2} l_2 + \ldots\ldots + C_{n,\,n-1} l_{n-1}] , \end{aligned}\right\} \tag{B}$$

where C_{1n} is the coefficient of $l_1 l_2 \ldots\ldots l_{n-1}$ in the product $\alpha_1 \alpha_2 \ldots\ldots \alpha_{n-1}$; but this is plainly

$$\begin{vmatrix} x_{11}, & x_{12}, & x_{13}, & \ldots\ldots, & x_{1,\,n-1} \\ x_{21}, & x_{22}, & x_{23}, & \ldots\ldots, & x_{2,\,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n-1,\,1}, & x_{n-1,\,2}, & x_{n-1,\,3}, & \ldots\ldots, & x_{n-1,\,n-1} \end{vmatrix} ,$$

and C_{11} is formed from this by dropping the left-hand column and adding on the right the next in cyclic order, and so on for the other C's in a perfectly evident manner. That is, C_{1n} is the minor of the determinant

$$\begin{vmatrix} x_{11}, & x_{12}, & x_{13}, & \ldots\ldots, & x_{1n} \\ x_{21}, & x_{22}, & x_{23}, & \ldots\ldots, & x_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n1}, & x_{n2}, & x_{n3}, & \ldots\ldots, & x_{nn} \end{vmatrix} = D ,$$

complementary to x_{nn}, C_{11} to x_{n1}, C_{12} to x_{n2}, and so on, but plainly the arrangement is such that

$$\begin{aligned} &x_{nn}C_{1n} + x_{n1}C_{11} + x_{n2}C_{12} + x_{n3}C_{13} + \ldots . + x_{n,\,n-1}C_{1,\,n-1} = D, \\ &x_{\mu n}C_{1n} + x_{\mu 1}C_{11} + x_{\mu 2}C_{12} \qquad\qquad + \ldots . + x_{\mu,\,n-1}C_{1,\,n-1} = 0. \end{aligned}$$

Also we have plainly in the case of $\omega_{n-1}(\alpha_\lambda)$,

$$\begin{aligned} &x_{\lambda-1,\,n}C_{\lambda n} + x_{\lambda-1,\,1}C_{\lambda 1} + x_{\lambda-1,\,2}C_{\lambda 2} + x_{\lambda-1,\,3}C_{\lambda 3} + \ldots . + x_{\lambda-1,\,n-1}C_{\lambda,\,n-1} = D, \\ &x_{\mu n}C_{\lambda n} + x_{\mu 1}C_{\lambda 1} + x_{\mu 2}C_{\lambda 2} + \ldots\ldots\ldots\ldots\ldots\ldots + x_{\mu,\,n-1}C_{\lambda,\,n-1} \quad = 0, \end{aligned}$$

μ being different from $\lambda - 1$.

We have also

$$\begin{aligned} &x_{nn}C_{1n} + x_{1n}C_{2n} + x_{2n}C_{3n} + \ldots . + x_{n-1,\,n}C_{nn} = D, \\ &x_{n\mu}C_{1n} + x_{1\mu}C_{2n} + x_{2\mu}C_{3n} + \ldots . + x_{n-1,\,\mu}C_{nn} = 0. \end{aligned}$$

All these equations are immediate consequences of the definition of a determinant combined with the equation

$$\omega_n\alpha_1\alpha_2 \ldots . \alpha_\kappa\alpha_\kappa \ldots . = 0.$$

Now making

$$\begin{array}{llll} z_{11} = x_{nn}, & z_{12} = x_{1n}, & z_{13} = x_{2n}, & \ldots . ,\ z_{1n} = x_{n-1,\,n}, \\ z_{21} = x_{n1}, & z_{22} = x_{11}, & z_{23} = x_{21}, & \ldots . ,\ z_{2n} = x_{n-1,\,1}, \\ z_{31} = x_{n2}, & z_{32} = x_{12}, & z_{33} = x_{22}, & \ldots . ,\ z_{3n} = x_{n-1,\,2}, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\ z_{\mu 1} = x_{n,\,\mu-1}, & z_{\mu 2} = x_{1,\,\mu-1}, & z_{\mu 3} = x_{2,\,\mu-1} & \ldots . ,\ z_{\mu n} = x_{n-1,\,\mu-1}, \end{array}$$

there results

$$\begin{aligned} -\frac{1}{A}h_1 = {} & [x_{nn}C_{1n} + x_{1n}C_{2n} + x_{2n}C_{3n} + \ldots .]\, l_n \\ & + [x_{nn}C_{11} + x_{1n}C_{21} + x_{2n}C_{31} + \ldots .]\, l_1 + \ldots . , \end{aligned}$$

or

$$\left.\begin{aligned} h_1 &= -ADl_n, \\ h_2 &= -ADl_1, \\ h_3 &= -ADl_2, \\ &\ldots\ldots\ldots \\ h_n &= -ADl_{n-1}. \end{aligned}\right\} \qquad \text{(C)}$$

It is plain enough now that ω_{n-1}^n does not differ in meaning from ω_n multiplied by $(-A)^n$. From the group (B) we have

$$\omega_{n-1}^n[\omega_{n-1}(\alpha_1)\,\omega_{n-1}(\alpha_2)\ldots\omega_{n-1}(\alpha_n)] = (-A)^n \begin{vmatrix} C_{1n}, & C_{11}, & C_{12}, & C_{13}, \ldots . & C_{1,\,n-1} \\ C_{2n}, & C_{21}, & C_{22}, & C_{23}, \ldots . & C_{n,\,n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ C_{nn}, & C_{n1}, & C_{n2}, & C_{n3}, \ldots . & C_{n,\,n-1} \end{vmatrix} l_n l_1 l_2 \ldots . l_{n-1} \qquad (9)$$

and from the group (C)

$$\omega_n h_1 h_2 \ldots . h_n = (-A)^n D^n l_n l_1 \ldots . l_{n-1}.$$

Substituting in equation (8) and replacing the z_{11}, z_{12} . . . . by their values,

$$(-A)^n D^n . l_n l_1 \dots . l_{n-1}$$

$$= \begin{vmatrix} x_{nn}, & x_{1n}, & x_{2n}, & \dots x_{n-1,n} \\ x_{n1}, & x_{11}, & x_{21}, & \dots x_{n-1,1} \\ \dots & \dots & \dots & \dots \\ x_{n,n-1}, & x_{1,n-1}, & x_{2,n-1}, & \dots x_{n-1,n-1} \end{vmatrix} \times \begin{vmatrix} C_{11}, & C_{12}, & C_{13}, & \dots C_{1n} \\ C_{21}, & C_{22}, & C_{23}, & \dots C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1}, & C_{n2}, & C_{n3}, & \dots . C_{nn} \end{vmatrix} (-A)^n l_n l_1 \dots l_{n-1}.$$

Hence

$$\begin{vmatrix} C_{11}, & C_{12}, & C_{13}, & \dots . C_{1n} \\ C_{21}, & C_{22}, & C_{23}, & \dots . C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1}, & C_{n2}, & C_{n3}, & \dots . C_{nn} \end{vmatrix} = D^{n-1},$$

a well known theorem.

Let it be required to determine $x_1, x_2, \dots . x_n$ in the equation

$$\rho = x_1\omega_{n-1}(\alpha_1) + x_2\omega_{n-1}(\alpha_2) + \dots . + x_n\,\omega_{n-1}(\alpha_n).$$

We have

$$S\alpha_n\rho = x_1 S.[\omega_{n-1}(\alpha_1)]\,\alpha_n + \dots . + x_\mu S.[\omega_{n-1}(\alpha_\mu)]\,\alpha_n + \dots .$$

We have seen that

$$\omega_{n-1}(\alpha_1) = -A\,[C_{11}l_1 + C_{12}l_2 + \dots . + C_{1n}l_n],$$

and also

$$\alpha_n = x_{n1}l_1 + x_{n2}l_2 \qquad + \dots . + x_{nn}l_n.$$

Hence

$$S.[\omega_{n-1}(\alpha_1)]\,\alpha_n = A\,[x_{n1}C_{11} + x_{n2}C_{12} + \dots . + x_{nn}C_{1n}] = AD = \omega_n . \alpha_1\alpha_2 \dots . \alpha_n.$$

Also

$$S.[\omega_{n-1}(\alpha_\mu)]\,\alpha_n = A\,[x_{n1}C_{\mu 1} + x_{n2}C_{\mu 2} + \dots . + x_{nn}C_{\mu n}] = 0.$$

Therefore

$$x_1 = \frac{1}{AD}S\alpha_n\rho, \quad x_2 = \frac{1}{AD}S\alpha_1\rho, \quad x_\mu = \frac{1}{AD}S\alpha_{\mu-1}\rho.$$

These results may be elegantly reached as follows, viz: The product of α_n into $\omega_{n-1}(\alpha_1)$ is made up of products of l_μ into terms which do contain l_μ and terms which do not contain l_μ but contain the other $n-1$ units. The former products are all vectors; the latter are all scalars and when $\mu = 1, 2, \dots . n$ their aggregate equals $\omega_n . \alpha_1\alpha_2 \dots . \alpha_n$ by definition. Hence $S.[\omega_{n-1}(\alpha_\mu)]\,\alpha_{\mu-1} = \omega_n(\alpha_\mu)$, and $S.[\omega_{n-1}(\alpha_\mu)]\,\alpha_\lambda = 0$ for $\omega_n . \alpha_\mu\alpha_{\mu+1} \dots . \alpha_\lambda \dots . \alpha_\lambda = 0$ as has been shown already. Hence

$$S.\alpha_{\mu-1}\rho = x_\mu\omega_n(\alpha_\mu) = x_\mu AD.$$

Hence

$$\left.\begin{aligned} AD\rho = \omega_{n-1}(\alpha_1)\,S\alpha_n\rho + \omega_{n-1}(\alpha_2)\,S\alpha_1\rho + \omega_{n-1}(\alpha_3)\,S\alpha_2\rho + \dots . \\ + \omega_{n-1}(\alpha_n).S\alpha_{n-1}\rho \end{aligned}\right\} \qquad (10)$$

Let $\beta_1, \beta_2, \beta_3, \dots . \beta_n$ be any n vectors, then

$$AD\beta_1 = \omega_{n-1}(\alpha_1).S\alpha_n\beta_1 + \omega_{n-1}(\alpha_2).S\alpha_1\beta_1 + \dots . + \omega_{n-1}(\alpha_n).S\alpha_{n-1}\beta_1,$$

. . . .

$$AD\beta_n = \omega_{n-1}(\alpha_1).S\alpha_n\beta_n + \omega_{n-1}(\alpha_2).S\alpha_1\beta_n + \dots . + \omega_{n-1}(\alpha_n).S\alpha_{n-1}\beta_n;$$

and applying ω_n,

$$\begin{aligned} A^n D^n \omega_n . \beta_1 \ldots\ldots \beta_n &= A^n D^n . AD' = A^{n+1} D^n D' \\ &= \omega^n_{n-1} [\omega_{n-1}(\alpha_1) . \omega_{n-1}(\alpha_2) \ldots\ldots \omega_{n-1}(\alpha_n)] \end{aligned} \begin{vmatrix} S\alpha_1\beta_1, & S\alpha_2\beta_1, & \ldots\ldots, & S\alpha_n\beta_1 \\ S\alpha_1\beta_2, & S\alpha_2\beta_2, & \ldots\ldots, & S\alpha_n\beta_2 \\ \ldots & \ldots & \ldots & \ldots \\ S\alpha_1\beta_n, & S\alpha_2\beta_n, & \ldots\ldots, & S\alpha_n\beta_n \end{vmatrix} .$$

But $\omega^n_{n-1} [\omega_{n-1}(\alpha_1) \omega_{n-1}(\alpha_2) \ldots\ldots \omega_{n-1}(\alpha_n)] = (-1)^n A^{n+1} D^{n-1}$ by eq. (9); and this is $-A^{n+1}D^{n-1}$, since n is odd. Therefore

$$DD' = - \begin{vmatrix} S\alpha_1\beta_1, & S\alpha_2\beta_1, & \ldots\ldots & S\alpha_n\beta_1 \\ \ldots & \ldots & \ldots & \ldots \\ S\alpha_1\beta_n, & S\alpha_2\beta_n, & \ldots\ldots & S\alpha_n\beta_n \end{vmatrix} ,$$

which is the rule for multiplication in another form. If the scalars be written out it takes the ordinary form.

Let $\phi\rho$ be a vector function into which ρ enters linearly; there will be terms of the form $\delta S\gamma_1\gamma_2 \ldots\ldots \gamma_r\rho$ and terms of the form $z\rho$, where z is a scalar. By aid of eq. (10), ρ in the terms of the form $z\rho$ may be expressed in terms of an arbitrary system of n vectors, $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$, provided D does not vanish. The vectors δ may be expressed in terms of the same system and $\phi\rho$ will take the form

$$\begin{aligned} \phi\rho = \omega_{n-1}(\alpha_1)[\Sigma S . \gamma_{11}\gamma_{12} \ldots\ldots \gamma_{1\mu}\rho] + \omega_{n-1}(\alpha_2)[\Sigma S . \gamma_{21}\gamma_{22} \ldots\ldots \gamma_{2\nu}\rho] + \ldots\ldots \\ + \omega_{n-1}(\alpha_n)[\Sigma S . \gamma_{n1}\gamma_{n2} \ldots\ldots \gamma_{n\tau}\rho] . \end{aligned}$$

But $\Sigma S . \gamma_1\gamma_2 \ldots\ldots \gamma_\mu\rho = S . [\Sigma\gamma_1\gamma_2 \ldots\ldots \gamma_\mu]\,\rho = S\beta_1\rho$. Hence we have finally

$$\phi\rho = \omega_{n-1}(\alpha_1) . S\beta_1\rho + \omega_{n-1}(\alpha_2) . S\beta_2\rho + \ldots\ldots + \omega_{n-1}(\alpha_n) . S\beta_n\rho = \gamma . \qquad (11)$$

Eq. 11 being given, let it be required to express $\phi^{-1}\gamma = \rho$ in terms of the known operator ϕ. For brevity, write $\phi\rho$ as follows, viz:

$$\phi\rho = \delta_1 S\beta_1\rho + \delta_2 S\beta_2\rho + \ldots\ldots + \delta_n S\beta_n\rho ,$$

and take

$$\phi'\rho_1 = \beta_1 S\delta_1\rho_1 + \beta_2 S\delta_2\rho_1 + \ldots\ldots + \beta_n S\delta_n\rho_1 .$$

Then, ρ_1 being any vector whatever,

$$S\rho_1\phi\rho = S\rho\phi'\rho_1, \text{ evidently.}$$

Following Hamilton, on whose work all this is based, the functions ϕ and ϕ' may be called conjugate to each other. Now take $n-1$ vectors, $\lambda_1, \lambda_2, \ldots\ldots \lambda_{n-1}$ such that $\omega_{n-1} . \lambda_1\lambda_2 \ldots\ldots \lambda_{n-1} = \gamma = \phi\rho$, it is clear that

$$\begin{aligned} S\lambda_1\phi\rho &= S . \lambda_1\omega_{n-1} . \lambda_1\lambda_2 \ldots\ldots \lambda_{n-1} &&= \omega_n . \lambda_1\lambda_1\lambda_2 \ldots\ldots \lambda_{n-1} &&= 0 , \\ S\lambda_\kappa\phi\rho &= S . \lambda_\kappa\omega_{n-1} . \lambda_1\lambda_2 \ldots\ldots \lambda_\kappa \ldots\ldots \lambda_{n-1} &&= \omega_n . \lambda_\kappa\lambda_1\lambda_2 \ldots\ldots \lambda_\kappa \ldots\ldots \lambda_{n-1} &&= 0 , \end{aligned}$$

where $\mu = 1, 2, \ldots . n - 1$. But

$$S\lambda_\kappa\phi\rho = S\rho\phi'\lambda_\kappa = 0,$$

giving $n - 1$ conditions which will all be fulfilled if

$$m\rho = \omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_{n-1},$$

m being a scalar whose value is to be found.

But since $\phi\rho = \omega_{n-1}.\lambda_1\lambda_2 \ldots . \lambda_{n-1}$,

$$\rho = \phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \ldots . \lambda_{n-1}];$$

hence,

$$m\phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \ldots . \lambda_{n-1}] = \omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_{n-1}. \tag{12}$$

Let λ_n be a vector such that $\omega_n.\lambda_1\lambda_2 \ldots . \lambda_n \neq 0$; then

$$S.\phi'\lambda_n\phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \ldots . \lambda_{n-1}] = S.\lambda_n\phi\phi^{-1}[\omega_{n-1}.\lambda_1\lambda_2 \ldots . \lambda_{n-1}]$$
$$= S.\lambda_n[\omega_{n-1}.\lambda_1 \ldots . \lambda_{n-1}] = S.[\omega_{n-1}.\lambda_1\lambda_2 \ldots . \lambda_{n-1}]\lambda_n = \omega_n.\lambda_1\lambda_2 \ldots . \lambda_n.$$

Also

$$S.\phi'\lambda_n[\omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_{n-1}] = S.[\omega_{n-1}.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_{n-1}]\phi'\lambda_n$$
$$= \omega_n.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_n.$$

Therefore, multiplying eq. (12) by $\phi'\lambda_n$ and taking scalars, m is given by the equation

$$m = \frac{\omega_n.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_n}{\omega_n.\lambda_1\lambda_2 \ldots . \lambda_n}$$

As m is a homogeneous function of degree 0 in $\lambda_1, \lambda_2, \ldots . \lambda_n$, it is not altered by linear transformation; its value is therefore independent of the particular auxiliary system chosen in any given case. This is easy to see, viz: taking

$$\lambda_1 = c_{11}l_1 + c_{12}l_2 + \ldots . + c_{1n}l_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_\mu = c_{\mu 1}l_1 + c_{\mu 2}l_2 + \ldots . + c_{\mu n}l_n,$$

we have

$$\omega_n.\lambda_1 \ldots . \lambda_n = \begin{vmatrix} c_{11}, & c_{12}, & \ldots . & c_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n1}, & c_{n2}, & \ldots . & c_{nn} \end{vmatrix} A = CA.$$

Also,

$$\phi'\lambda_1 = c_{11}\phi'l_1 + c_{12}\phi'l_2 + \ldots . + c_{1n}\phi'l_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi'\lambda_\mu = c_{\mu 1}\phi'l_1 + c_{\mu 2}\phi'l_2 + \ldots . + c_{nn}\phi'l_n,$$

and

$$\omega_n.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_n = C\omega_n.\phi'l_1\phi'l_2 \ldots . \phi'l_n.$$

Hence

$$m = \frac{\omega_n.\phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_n}{\omega_n.\lambda_1\lambda_2 \ldots . \lambda_n} = \frac{1}{A}\omega_n.\phi'l_1\phi'l_2 \ldots . \phi'l_n.$$

Making $\phi = \phi + g$ and therefore $\phi' = \phi' + g$, and repeating the calculation of m for the new function, it comes out

$$\begin{aligned} mg &= \frac{\omega_n . (\phi' + g)\lambda_1 (\phi' + g)\lambda_2 \ldots . (\phi' + g)\lambda_n}{\omega_n . \lambda_1\lambda_2 \ldots . \lambda_n} \\ &= \frac{1}{A}\,\omega_n . (\phi' + g) l_1 (\phi' + g) l_2 \ldots . (\phi' + g) l_n . \end{aligned}$$

That is,

$$mg = m + m_1 g + m_2 g^2 + \ldots . + m_\mu g^\mu + \ldots . + m_n g^n,$$

the values of m_1, $m_2 \ldots .$ being evident, and in particular, $m_n = 1$. From eq. (12),

$$\begin{aligned} & mg\,(\phi + g)^{-1}[\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1}] \\ &= \omega_{n-1} . [(\phi' + g)\lambda_1 (\phi' + g)\lambda_2 \ldots . (\phi' + g)\lambda_{n-1}] \\ &= \omega_{n-1} . \phi'\lambda_1\phi'\lambda_2 \ldots . \phi'\lambda_{n-1} + g[A] + g^2[B] + \ldots . + g^{n-1}\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1} \\ &= m\phi^{-1}[\omega_{n-1} . \lambda_1\lambda_2 \ldots \lambda_{n-1}] + g[A] + g^2[B] + \ldots . + g^{n-1}\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1} . \end{aligned}$$

Operating on both sides with $(\phi + g)$,

$$\begin{aligned} & [m + m_1 g + m_2 g^2 + \ldots . + m_\mu g^\mu + \ldots . m_n g^n]\,\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1} \\ &= mg\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1} = m\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1} + g\,\{\phi[A] \\ &\quad + m\phi^{-1}[\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1}]\} + g^2\{\phi[B] + [A]\} \\ &\quad + g^3\{\phi[C] + [B]\} + \ldots . + g^{n-2}\{\phi[M] + [L]\} \\ &\quad + g^{n-1}\{\phi\,[\omega_{n-1} . \lambda_1 \ldots . \lambda_{n-1}] + [M]\} + g^n\omega_{n-1} . \lambda_1\lambda_2 \ldots . \lambda_{n-1} . \end{aligned}$$

Now g being any scalar whatever, its coefficients on both sides are equal, and this is assured by noting that the coefficients of g^ν and g^n are identical, since $m_n = 1$. Hence

$$\phi[A] + m\phi^{-1}\omega_{n-1}(\lambda_1) = m_1\omega_{n-1}(\lambda_1),$$

or

$$[A] = m_1\phi^{-1}\omega_{n-1}(\lambda_1) - m\phi^{-2}\omega_{n-1}(\lambda_1),$$

that is, $[A]$ is a linear and vector function of $\omega_{n-1} . (\lambda_1)$, and the same may be shown for $[B], \ldots .$ But we have

$$\begin{aligned} \phi\omega_{n-1}(\lambda) + [M] &= m_{n-1}\omega_{n-1}(\lambda_1), \\ \phi[M] + [L] &= m_{n-2}\omega_{n-1}(\lambda_1), \\ \phi[L] + [K] &= m_{n-3}\omega_{n-1}(\lambda_1), \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \phi[C] + [B] &= m_3\omega_{n-1}(\lambda_1), \\ \phi[B] + [A] &= m_2\omega_{n-1}(\lambda_1), \\ \phi[A] + m\phi^{-1}\omega_{n-1}(\lambda_1) &= m_1\omega_{n-1}(\lambda_1). \end{aligned}$$

Hence

$$\begin{aligned}
[M] &= [m_{n-1} - \phi]\,\omega_{n-1}(\lambda_1),\\
[L] &= [m_{n-2} - m_{n-1}\phi + \phi^2]\,\omega_{n-1}(\lambda_1),\\
[K] &= [m_{n-3} - m_{n-2}\phi + m_{n-1}\phi^2 - \phi^3]\,\omega_{n-1}(\lambda_1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
[A] &= [m_2 - m_3\phi + m_4\phi^2 - m_5\phi^3 + \ldots\ldots + m_{n-1}\phi^{n-3} - \phi^{n-2}]\,\omega_{n-1}(\lambda_1),\\
m\phi^{-1}\omega_{n-1}(\lambda_1) &= [m_1 - m_2\phi + m_3\phi^2 - \ldots\ldots - m_{n-1}\phi^{n-2} + \phi^{n-1}]\,\omega_{n-1}(\lambda_1).
\end{aligned}$$

$\omega_{n-1}(\lambda_1)$ is an arbitrary vector; hence if it be omitted there will remain a symbolical equation in ϕ true for any vector that may be inserted, therefore

$$m\phi^{-1} = m_1 - m_2\phi + m_3\phi^2 - \ldots\ldots - m_{n-1}\phi^{n-2} + \phi^{n-1}, \tag{12}$$

or $$\phi^n - m_{n-1}\phi^{n-1} + m_{n-2}\phi^{n-2} - \ldots\ldots + m_3\phi^3 - m_2\phi^2 + m_1\phi - m = 0. \tag{13}$$

The coefficients $m_1, m_2, \ldots\ldots m_{n-1}$ are all independent of the vectors $\lambda_1 \ldots\ldots \lambda_n$, for they can be expressed in terms of the units $l_1 \ldots\ldots l_n$ and the constant vectors of ϕ'.

Equation (13) may be written

$$(\phi + g_1)(\phi + g_2)\ldots\ldots(\phi + g_n) = 0,$$

which means that n vectors $\delta_1, \delta_2, \ldots\ldots \delta_n$ can be found such that each of them satisfies a relation of the form

$$(\phi + g_\kappa)\,\delta_\kappa = 0, \text{ or } \phi\delta_\kappa = -g_\kappa\delta_\kappa.$$

But plainly $g_1, g_2, \ldots\ldots g_n$ are the roots of $m_g = 0$. Of course there is a relation of the same kind for ϕ', viz:

$$(\phi' + h_1)(\phi' + h_2)\ldots\ldots(\phi' + h_n) = 0,$$

where $h_1 \ldots\ldots h_n$ are the roots of an equation

$$m_h = 0,$$

m_h being the quantity related to ϕ' as m_g is to ϕ. The operation $(\phi + g)^{-1}$ has already been determined as follows, viz:

$$m_g(\phi + g)^{-1}[\omega_{n-1}.\lambda_1 \ldots\ldots \lambda_{n-1}] = \omega_{n-1}.\,[(\phi' + g)\,\lambda_1\,(\phi' + g)\,\lambda_2 \ldots\ldots (\phi' + g)\,\lambda_{n-1}]$$

and for $(\phi' + h)^{-1}$ we have therefore

$$m_h(\phi' + h)^{-1}[\omega_{n-1}.\lambda_1 \ldots\ldots \lambda_{n-1}] = \omega_{n-1}.\,[(\phi + h)\,\lambda_1(\phi + h)\,\lambda_2 \ldots\ldots (\phi + h)\,\lambda_{n-1}],$$

where $$m_h = \frac{1}{A}\,\omega_n.(\phi + h)\,l_1(\phi + h)\,l_2 \ldots\ldots (\phi + h)\,l_n.$$

Let

$$D^n\rho + P_1D^{n-1}\rho + P_2D^{n-2}\rho + \ldots\ldots + P_n\rho = 0, \text{ where } D^\mu = \frac{d^\mu}{dt^\mu}, \tag{18}$$

be a linear differential equation of the n^{th} order in ρ with uniform coefficients, and t be the independent scalar variable; then, if

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots\ldots + x_n\alpha_n$$

be an integral of (18), $x_1, x_2, \ldots\ldots x_n$ are a system of integrals of the scalar equation

$$D^n x + P_1 D^{n-1} x + \ldots\ldots + P_n x = 0, \tag{19}$$

provided $\omega_n . \alpha_1\alpha_2 \ldots\ldots \alpha_n$ does not vanish. For substituting the value of ρ in eq. (18), the coefficients of $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ must separately vanish; but these coefficients are the results of substituting $x_1, x_2, \ldots\ldots x_n$ successively in eq. (19). Evidently, also, if $x_1, x_2, \ldots\ldots x_n$ are a system of independent integrals of eq. (19), the equation in ρ will be satisfied by

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots\ldots + x_n\alpha_n,$$

$\alpha_1, \ldots\ldots \alpha_n$ being any constant vectors whatever.

If ρ be expressed in terms of any other set of independent vectors as $\beta_1, \beta_2, \ldots\ldots \beta_n$, the scalar coefficients will still be integrals of (19). For we have, changing the notation of eq. (10),

$$\alpha_1 = c_{11}\beta_1 + c_{12}\beta_2 + \ldots\ldots + c_{1n}\beta_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_n = c_{n1}\beta_1 + c_{n2}\beta_2 + \ldots\ldots + c_{nn}\beta_n,$$

whence

$$\rho = (c_{11}x_1 + c_{21}x_2 + \ldots\ldots + c_{n1}x_n)\beta_1 + \ldots\ldots + (c_{1n}x_1 + c_{2n}x_2 + \ldots\ldots + c_{nn}x_n)\beta_n.$$

If ρ be an integral of eq. (18), $\phi\rho$ will also be an integral. For, evidently,

$$\phi\,[D^n\rho + P_1 D^{n-1}\rho + \ldots\ldots + P_n\rho] = 0.$$

But ϕ is commutative with D as is well known, hence

$$D^n\phi\rho + P_1 D^{n-1}\phi\rho + \ldots\ldots + P_n D\rho = 0.$$

This is also clear from the fact that $\phi\rho$ is obtained from ρ by a linear substitution. For, taking

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots\ldots + x_n\alpha_n,$$

and
$$\phi\rho = \alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho + \ldots\ldots + \alpha_n S\beta_n\rho,$$

we have
$$\phi\alpha_1 = \alpha_1 S\beta_1\alpha_1 + \alpha_2 S\beta_2\alpha_1 + \ldots\ldots + \alpha_n S\beta_n\alpha_1,$$

or
$$\phi\alpha_1 = a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots\ldots + a_{1n}\alpha_n,$$

and in the same way,

$$\phi\alpha\mu = a_{\mu 1}\alpha_1 + a_{\mu 2}\alpha_2 + \ldots\ldots + a_{\mu n}\alpha_n.$$

But
$$\phi\rho = x_1\phi\alpha_1 + x_2\phi\alpha_2 + \ldots\ldots + x_n\phi\alpha_n;$$

hence

$$\phi\rho = (a_{11}x_1 + a_{21}x_2 + \ldots\ldots + a_{n1}x_n)\,\alpha_1 + (a_{12}x_1 + a_{22}x_2 + \ldots\ldots + a_{n2}x_n)\,\alpha_2$$
$$+ \ldots\ldots + (a_{1n}x_1 + a_{2n}x_2 + \ldots\ldots + a_{nn}x\ )\,\alpha_n. \tag{20}$$

So that ρ may be transformed by changing the vectors of reference or by applying ϕ, and the coefficients will remain integrals of eq. (19). If

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$$

is an integral of eq. (18), it follows that $S\delta\rho = x_1 S\delta\alpha_1 + x_2 S\delta\alpha_2 + \dots + x_n S\delta\alpha_n$ is a general integral of eq. (19); and the same is true of $S.\delta\phi\rho$. Now let it be required to determine δ in such a manner that $S.\delta\phi\rho = hS\delta\rho$, h being a constant scalar; that is, to determine an integral of eq. (19) which is unaltered by a certain linear transformation, except as to a constant factor. We have

$$S.\delta\phi\rho = hS\delta\rho;\ \text{but}\ S.\delta\phi\rho = S\rho\phi'\delta,$$

hence

$$hS\delta\rho = S\rho\phi'\delta,\ \text{or}\ S.\rho(h\delta - \phi'\delta) = 0$$

whatever be ρ; but this requires that $\phi'\delta = h\delta$, hence h must be a root of the equation in ϕ' corresponding to the equation (13) in ϕ, and δ must be the corresponding vector. There are, therefore, n integrals of eq. (19) which are only multiplied by a constant by the transformation ϕ. If

$$S.\delta\phi\rho = hS\delta\rho,\ \text{then}\ S.\delta\phi^2\rho = h^2 S\delta\rho.$$

For

$$S.\delta\phi^2\rho = S.\phi'\delta\phi\rho = S.\phi'^2\delta\rho = S.\rho\phi'^2\delta = S.\rho h^2\delta = h^2 S\rho\delta.$$

Hence the integrals $S\delta_1\rho$, $S\delta_2\rho$, ... $S\delta_n\rho$ determined by the above process are independent, as may be seen in the usual way.

The above simple process may be varied as follows, viz: We have

$$\begin{aligned} S.\delta\phi\rho &= (a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n)\, S\delta\alpha_1 + \dots + (a_{1n}x_1 + a_{2n}x_2 + \dots a_{nn}x_n)\, S\delta\alpha_n \\ &= hS\delta\rho = h\,(x_1 S\delta\alpha_1 + x_2 S\delta\alpha_2 + \dots + x_n S\delta\alpha_n); \end{aligned}$$

giving n equations in h, viz:

$$\begin{aligned} (a_{11} - h)\, S\delta\alpha_1 + a_{12} S\delta\alpha_2 \qquad &+ \dots + a_{1n} S\delta\alpha_n = 0, \\ a_{21} S\delta\alpha_1 + (a_{22} - h)\, S\delta\alpha_2 &+ \dots + a_{2n} S\delta\alpha_n = 0, \\ \dots\dots\dots\dots\dots\dots&\dots\dots\dots\dots \\ a_{n1} S\delta\alpha_1 + a_{n2} S\delta\alpha_2 \qquad &+ \dots + a_{nn} S\delta\alpha_n = 0; \end{aligned}$$

whence h must be a root of the equation

$$\begin{vmatrix} a_{11} - h, & a_{12}, & a_{13}, & \dots, & a_{1n} \\ a_{21}, & a_{22} - h, & a_{23}, & \dots, & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & a_{n3}, & \dots, & a_{nn} - h \end{vmatrix} = 0. \tag{21}$$

If h_1 be a root of eq. (21), it may be substituted in the preceding equations, and the result is n conditions to determine δ_1; hence the statement that δ_1, δ_2, ... δ_n are determinate vectors is fully justified.

We know that the roots of eq. (21) are identical with those of

$$m_h = 0, \tag{22}$$

therefore the coefficients of the successive powers of h in eq. (21) are invariants. This fact is otherwise sufficiently clear, for those coefficients depend only upon certain constant vectors and not at all upon the variables in ρ.

In case the eq. $$m_h = 0$$

has a pair of equal roots, the properties of the corresponding integrals may be obtained with great ease, viz: It is clear that δ_1 may be so taken as to satisfy the relation $$S\delta_1\phi\rho = S\rho\phi'\delta_1 = h_1 S\rho\delta_1,$$

—h_1 being one of the equal roots. Then taking γ_2 so as to satisfy the equation

$$\phi'\gamma_2 = \delta_1 + h_1\gamma_2, \tag{23}$$

we have $$S\rho\phi'\gamma_2 = S\gamma_2\phi\rho = S\delta_1\rho + h_1 S\gamma_2\rho.$$

If more than two roots of eq. (22) are equal, let their number be λ and let the vectors $\gamma_2, \gamma_3, \ldots . \gamma_\lambda$ be given by the following equations, viz:

$$\begin{aligned} \phi'\gamma_2 &= \delta_1 + h_1\gamma_2, \\ \phi'\gamma_3 &= \delta_1 + \gamma_2 + h_1\gamma_3, \\ \phi'\gamma_4 &= \delta_1 + \gamma_2 + \gamma_3 + h_1\gamma_4, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi'\gamma_\lambda &= \delta_1 + \gamma_2 + \gamma_3 + \ldots . + \gamma_{\lambda-1} + h_1\gamma_\lambda. \end{aligned}$$

Then the integrals and their forms after the substitution ϕ will be

$$\left.\begin{aligned} &S\delta_1\rho\,;\ S\delta_1\phi\rho = h_1 S\delta_1\rho, \\ &S\gamma_2\rho;\ S\gamma_2\phi\rho = S\delta_1\rho + h_1 S\gamma_2\rho, \\ &S\gamma_3\rho;\ S\gamma_3\phi\rho = S\delta_1\rho + S\gamma_2\rho + h_1 S\gamma_3\rho, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &S\gamma_\lambda\rho;\ S\gamma_\lambda\phi\rho = S\delta_1\rho + S\gamma_2\rho + S\gamma_3\rho + \ldots . + h_1 S\gamma_\lambda\rho; \end{aligned}\right\} \tag{24}$$

all these are obtained by multiplying by ρ and taking the scalars, remembering that $$S\rho\phi'\gamma = S\gamma\phi\rho.$$

It may be noted that γ_2 cannot be parallel to δ_1; for, making $\gamma_2 = z\delta_1$ and substituting in eq. (23), we find

$$z\phi'\delta_1 = \delta_1 + h_1 z\delta_1,$$

or $$h_1 z\delta_1 = \delta_1 + h_1 z\delta_1, \text{ or } \delta_1 = 0,$$

which is not in general true.

We may obtain the integrals in a much simpler form by taking

$$\begin{aligned} \phi'\gamma_2 &= h_1(\gamma_2 + \delta_1), \\ \phi'\gamma_3 &= h_1(\gamma_3 + \gamma_2), \\ &\dots\dots\dots\dots \\ \phi'\gamma_\lambda &= h_1(\gamma_\lambda + \gamma_{\lambda-1}), \end{aligned}$$

and the integrals before and after the substitution ϕ will be as follows, viz:

$$\left.\begin{aligned} &S\delta_1\rho\,; \quad S\delta_1\phi\rho = h_1 S\delta_1\rho, \\ &S\gamma_2\rho\,; \quad S\gamma_2\phi\rho = h_1[S\gamma_2\rho + S\delta_1\rho], \\ &S\gamma_3\rho\,; \quad S\gamma_3\phi\rho = h_1[S\gamma_3\rho + S\gamma_2\rho], \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &S\gamma_\lambda\rho\,; \quad S\gamma_\lambda\phi\rho = h_1[S\gamma_\lambda\rho + S\gamma_{\lambda-1}\rho]. \end{aligned}\right\} \tag{25}$$

The form of the system (25) is that obtained by Jordan (Cours d'Analyse, III, No. 144) by a somewhat complex analysis.

It remains to prove that the integrals of the system (25), together with these determined by the remaining vectors $\delta_{\lambda+1} \dots \delta_n$ which satisfy the equation

$$\phi'\delta_\kappa - h_\kappa\delta_\kappa = 0,$$

form an independent system. If they satisfy any linear relation, let it be the following:

$$c_1S\delta_1\rho + c_2S\gamma_2\rho + \dots + c_\lambda S\gamma_\lambda\rho + c_{\lambda+1}S\delta_{\lambda+1}\rho + \dots + c_nS\delta_n\rho = 0,$$

which may be written

$$S\rho.[c_1\delta_1 + c_2\gamma_2 + \dots + c_\lambda\gamma_\lambda + c_{\lambda+1}\delta_{\lambda+1} + \dots + c_n\delta_n] = 0 = S\rho\,[\eta] \tag{26}$$

$[\eta]$ representing the bracketed vectors. Eq. (26) is satisfied if ρ is perpendicular to $[\eta]$, which may happen only for certain directions; but, as t varies, ρ turns in an infinite number of directions; and it follows that eq. (26) cannot be true for all values of ρ unless $[\eta] = 0$. We must have then

$$c_1\delta_1 + c_2\gamma_2 + \dots + c_\lambda\gamma_\lambda + c_{\lambda+1}\delta_{\lambda+1} + \dots + c_n\delta_n = 0. \tag{27}$$

Operating on eq. (27) with

$$(\phi' - h_1)(\phi' - h_{\lambda+1})(\phi' - h_{\lambda+2}) \dots (\phi' - h_n),$$

the vectors $\delta_1, \delta_{\lambda+1}, \dots \delta_n$ all disappear, leaving the equation

$$(\phi' - h_1)(\phi' - h_{\lambda+1}) \dots (\phi' - h_n)[c_2\gamma_2 + \dots + c_\lambda\gamma_\lambda] = 0. \tag{28}$$

Eq. (26) may be satisfied in two ways: (1) if the vectors $\gamma_2 \dots \gamma_n$ are parallel respectively to any vectors of the system $\delta_1, \delta_{\lambda+1}, \dots \delta_n$; or (2) if $c_2\gamma_2 + \dots + c_\lambda\gamma_\lambda = 0$. But, referring back to the assumed values of $\gamma_2, \dots \gamma_\lambda$, we have

$$(\phi' - h_1)\gamma_2 = \delta_1, \tag{29}$$

and if $\gamma_2 - z\delta_1$, the first member vanishes leaving $\delta_1 = 0$, which is false. If again $\gamma_2 = z\delta_\kappa$, we have

$$z\phi'\delta_\kappa - zh_1\delta_\kappa = \delta_1$$

or

$$z(h_\kappa - h_1)\,\delta_\kappa = \delta_1;$$

but δ_κ cannot be parallel to δ_1 unless the roots h_κ and h_1 of eq. (22) are equal, as may be plainly seen from the remark immediately following eq. (21); and these roots are, by hypothesis, not equal. Hence γ_2 cannot be parallel to any of the vectors $\delta_1, \delta_{\lambda+1}, \ldots. \delta_n$. Moreover, we have

$$\phi'\gamma_\lambda - h_1\gamma_\lambda = h_1\gamma_{\lambda-1}. \tag{30}$$

Making $\gamma_\lambda = z\delta_\kappa$ in eq. (30), it becomes

$$z(h_\kappa - h_1)\,\delta_\kappa = h_1\gamma_{\lambda-1},$$

that is, if any vector γ_λ is parallel to δ_κ, the next preceding it is also parallel to δ_κ. This leads directly back to the assertion that γ_2 is parallel to δ_κ, which we know to be false. Therefore, if eq. (29) is true we must have

$$c_2\gamma_2 + \ldots. + c_\lambda\gamma_\lambda = 0.$$

This reduces eq. (27) to the following, viz:

$$c_1\delta_1 + c_{\lambda+1}\delta_{\lambda+1} + \ldots. + c_n\delta_n = 0. \tag{31}$$

Operating on eq. (31) with ϕ' successively $n - \lambda$ times, we have

$$\begin{aligned}
c_1\delta_1 + c_{\lambda+1}\delta_{\lambda+1} \qquad &+ \ldots. + c_n\delta_n &= 0,\\
h_1c_1\delta_1 + h_{\lambda+1}c_{\lambda+1}\delta_{\lambda+1} &+ \ldots. + h_nc_n\delta_n &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots&\ldots\ldots\ldots\ldots\ldots\ldots&\\
h_1^{n-\lambda}c_1\delta_1 + h_{\lambda+1}^{n-\lambda}c_{\lambda+1}\delta_{\lambda+1} &+ \ldots. + h_n^{n-\lambda}c_n\delta_n &= 0.
\end{aligned}$$

Eliminating $c_1\delta_1, c_{\lambda+1}\delta_{\lambda+1}, \ldots. c_n\delta_n$, it is found that the determinant

$$\begin{vmatrix}
1 & , 1 & , \ldots. , 1\\
h_1 & , h_{\lambda+1} & , \ldots. , h_n\\
h_1^2 & , h_{\lambda+1}^2 & , \ldots. , h_n^2\\
\ldots & \ldots & \ldots\\
h_1^{n-\lambda} & , h_{\lambda+1}^{n-\lambda} & , \ldots. , h_n^{n-\lambda}
\end{vmatrix}$$

must vanish. But the value of this determinant is the product of all the differences $(h_{\lambda+1} - h_1) \ldots.$, etc., and cannot vanish because no two of these roots are equal. Therefore there can be no linear relation among the integrals as determined above.

The operator ϕ represents the substitution which takes place among the independent integrals when t goes around a critical point and its constants necessarily change in passing from one critical point to another.